Mean and Variance of the Local Maxima of a Rayleigh Fading Envelope

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***Abstract*—A derivation is provided for the mean and variance of the local maxima of a Rayleigh fading envelope. The calculation is performed for a uniform 2-D angular distribution of incident power using the autocorrelation function given in the book by Jakes. This calculation is useful to obtain estimates of the mean received signal strength over a given spatial averaging interval.**

***Index Terms*—Probability, Rayleigh fading, statistics, wireless communication.**

I. Introduction

THE MEAN signal strength received at a mobile station in a wireless system is needed to determine the quality of the radio link for use in handoff, power control, and channel assignment algorithms. For Rayleigh fading with a uniform two-dimensional (2-D) angular distribution of incident power, a spatial averaging interval of 20 to 40 wavelengths has been suggested to estimate the mean received signal strength [1]. In practice, only signal samples taken at a constant temporal interval are available. The corresponding temporal averaging interval depends on the mobile speed. Since the local maxima of the received envelope occur with a characteristic spatial scale, a fixed number of local maxima can be used to estimate the received signal strength averaged over a given spatial interval for variable mobile speed in a manner similar to the method given in [2]. The mean of the local maxima is needed for this estimation method. In addition, the mean square error of the estimate is determined by the variance of the local maxima.

II. Signal Model

Let $r(x)$ denote the envelope received at a mobile station as a function of the scalar position variable x in a Rayleigh fading environment with a uniform 2-D angular distribution of incident power. The corresponding in-phase component, quadrature component, and phase of the received signal are denoted by $r_I(x)$, $r_Q(x)$, and $\theta(x)$, respectively. For these conditions, the autocorrelation of the in-phase (or quadrature) component is given by [3]

$$\begin{aligned} A_{r_I r_I}(\Delta x) &= E[r_I(x+\Delta x) r_I(x)] \\ &= A_{r_Q r_Q}(\Delta x) = b_0 J_0(2\pi \Delta x/\lambda) \end{aligned} \tag{1}$$

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where $b_0 = E[r_I^2(x)]$; $J_0(\cdot)$ is the zeroth-order Bessel function of the first kind, and λ is the carrier wavelength. We note that for this model, $r_I(x)$ and $r_Q(x)$ are zero-mean Gaussian random processes, and $\theta(x)$ is a zero-mean random process uniformly distributed in $[0, 2\pi)$.

III. Mean and Variance of Envelope Local Maxima

Let $\{r_k^{\max}\}$ denote the local maxima of $r(x)$. From [4], [5], the probability that the Rayleigh envelope $r(x)$ has a local maximum in the distance interval $(x, x+dx)$ and in the amplitude interval $(r, r+dr)$ is

$$-dx\, dr \int_{-\infty}^{0} \rho(r, r'=0, r'')\, r''\, dr'' \tag{2}$$

where r' and r'' denote the first and second derivatives of $r(x)$ with respect to x, and $\rho(r, r', r'')$ is the joint probability density function (pdf) of (r, r', r''). Therefore, the first and second moments of the local maxima are given by

$$E\left\{(r_k^{\max})^n\right\} = \frac{\displaystyle\int_{r=0}^{\infty}\int_{r''=-\infty}^{0} r^n\, r''\, \rho(r, r'=0, r'')\, dr''\, dr}{\displaystyle\int_{r=0}^{\infty}\int_{r''=-\infty}^{0} r''\, \rho(r, r'=0, r'')\, dr''\, dr} \tag{3}$$

where $n=1$ for the first moment and $n=2$ for the second moment.

The pdf $\rho(r, r', r'')$ is determined by the change of variables $(r_I, r_Q, r_I', r_Q', r_I'', r_Q'') \rightarrow (r, \theta, r', \theta', r'', \theta'')$. The Jacobian of this transformation is r^3. The vector $\mathbf{z} \equiv [r_I, r_Q, r_I', r_Q', r_I'', r_Q'']^T$ is a zero-mean Gaussian random vector with covariance

$$\Sigma = \begin{bmatrix} b_0 & 0 & 0 & 0 & -b_2 & 0 \\ 0 & b_0 & 0 & 0 & 0 & -b_2 \\ 0 & 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & 0 & 0 \\ -b_2 & 0 & 0 & 0 & b_4 & 0 \\ 0 & -b_2 & 0 & 0 & 0 & b_4 \end{bmatrix} \tag{4}$$

where $b_2 = (b_0/2)(2\pi/\lambda)^2$, and $b_4 = (3b_0/8)(2\pi/\lambda)^4$. The pdf of $\mathbf{z}$ is then

$$\rho(\mathbf{z}) = \frac{1}{(2\pi)^3 |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{z}^T \Sigma^{-1} \mathbf{z}\right). \tag{5}$$

The change of variables yields the pdf of $(r, \theta, r', \theta', r'', \theta'')$:

$$\rho(r, \theta, r', \theta', r'', \theta'') = \frac{r^3}{(2\pi)^3 b_2 (b_0 b_4 - b_2^2)} \exp\left\{-\frac{1}{2b_2(b_0 b_4 - b_2^2)} [b_2 b_4 r^2\right.$$

$$- b_2^2(r')^2 + b_0 b_4 (r')^2 + 2b_2^2 r r'' + b_0 b_2 (r'')^2 - 3b_2^2 r^2 (\theta')^2 + b_0 b_4 r^2 (\theta')^2 + 4b_0 b_2 (r')^2 (\theta')^2 - 2b_0 b_2 r r'' (\theta')^2 + b_0 b_2 r^2 (\theta')^4 + 4 b_0 b_2 r r' \theta' \theta'' + b_0 b_2 r^2 (\theta'')^2] \Big\}. \quad (6)$$

The pdf $\rho(r, r' = 0, r'')$ for $r'' > [(b_0 b_4 - 3b_2^2)/(2b_0 b_2)]r$ is

$$\begin{aligned}\rho(r, r' = 0, r'') &= \int_{\theta=0}^{2\pi} \int_{\theta'=-\infty}^{\infty} \int_{\theta''=-\infty}^{\infty} \times \rho(r, \theta, r' = 0, \theta', r'', \theta'')\, d\theta''\, d\theta'\, d\theta \\ &= \frac{r^{3/2}[(3b_2^2 - b_0 b_4) r + 2b_0 b_2 r'']^{1/2}}{8\sqrt{\pi}\, b_0 b_2^{3/2} (b_0 b_4 - b_2^2)^{1/2}} \\ &\quad \times \exp\left\{ \frac{[(3b_2^2 - b_0 b_4) r + 2b_0 b_2 r'']^2}{16 b_0 b_2^2 (b_0 b_4 - b_2^2)} - \frac{b_4 r^2 + 2b_2 r r'' + b_0 (r'')^2}{2(b_0 b_4 - b_2^2)} \right\} \\ &\quad \times \left[I_{-1/4}\left(\frac{[(3b_2^2 - b_0 b_4) r + 2b_0 b_2 r'']^2}{16 b_0 b_2^2 (b_0 b_4 - b_2^2)} \right) + I_{1/4}\left(\frac{[(3b_2^2 - b_0 b_4) r + 2b_0 b_2 r'']^2}{16 b_0 b_2^2 (b_0 b_4 - b_2^2)} \right) \right] \end{aligned} \quad (7)$$

where $I_{\pm 1/4}(\cdot)$ is the modified Bessel function of the first kind of order $\pm 1/4$. For $r'' \leq [(b_0 b_4 - 3b_2^2)/(2b_0 b_2)]r$

$$\begin{aligned}\rho(r, r' = 0, r'') &= \frac{r^{3/2}[-(3b_2^2 - b_0 b_4) r - 2b_0 b_2 r'']^{1/2}}{\sqrt{32\pi^3}\, b_0 b_2^{3/2} (b_0 b_4 - b_2^2)^{1/2}} \\ &\quad \times \exp\left\{ \frac{[(3b_2^2 - b_0 b_4) r + 2b_0 b_2 r'']^2}{16 b_0 b_2^2 (b_0 b_4 - b_2^2)} - \frac{b_4 r^2 + 2b_2 r r'' + b_0 (r'')^2}{2(b_0 b_4 - b_2^2)} \right\} \\ &\quad \times K_{1/4}\left(\frac{[(3b_2^2 - b_0 b_4) r + 2b_0 b_2 r'']^2}{16 b_0 b_2^2 (b_0 b_4 - b_2^2)} \right) \end{aligned} \quad (8)$$

where $K_{1/4}(\cdot)$ is the modified Bessel function of the second kind of order 1/4.

After substitution of the expressions for b_0, b_2, b_4 and the change of variables $x = r''/[\sqrt{b_0}(2\pi/\lambda)^2]$, $y = r/\sqrt{b_0}$, we obtain the following integrals necessary for evaluating (3)

$$\begin{aligned}\mathbf{I}_n &= b_0^{n/2} \int_{y=0}^{\infty} \int_{x=-\infty}^{-(3y/8)} \tfrac{1}{2} x y^{n+(3/2)} (-3y/8 - x)^{1/2} \\ &\quad \times e^{2(3y/8+x)^2 - 3y^2/2 - 4xy - 4x^2} \\ &\quad \times K_{1/4}\left(2(3y/8 + x)^2\right)\, dx\, dy \\ &\quad + b_0^{n/2} \int_{y=0}^{\infty} \int_{x=-(3y/8)}^{0} \\ &\quad \times \frac{\pi}{2\sqrt{2}} x y^{n+(3/2)} (3y/8 + x)^{1/2} \\ &\quad \times e^{2(3y/8+x)^2 - 3y^2/2 - 4xy - 4x^2} \\ &\quad \times \left[I_{-1/4}(2(3y/8 + x)^2) + I_{1/4}(2(3y/8 + x)^2) \right]\, dx\, dy. \end{aligned} \quad (9)$$

The mean and variance of $r_k^{\max}$ are determined by numerical integration

$$E[r_k^{\max}] = \frac{\mathbf{I}_1}{\mathbf{I}_0} \approx 1.72\sqrt{b_0} \quad (10)$$

$$\operatorname{var}[r_k^{\max}] = \frac{\mathbf{I}_2}{\mathbf{I}_0} - E^2[r_k^{\max}] \approx 0.46\, b_0. \quad (11)$$

Results of computer simulation agree with the values given above.

IV. Conclusion

A calculation is presented to determine the mean and variance of the local maxima of a Rayleigh fading envelope. The mean of the local maxima is useful to obtain an estimate of the received signal strength averaged over a given spatial interval. The mean square error of the estimate can be computed using the variance of the local maxima. The analytical values of the mean and variance of the local maxima are validated by computer simulation.

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