Channel and Capacity Estimation Errors

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***Abstract*—Systems with multiple element transmitter and receiver arrays have been shown to achieve very high spectral efficiencies. The theoretically achievable Shannon capacity is a function of the channel between the transmitters and the receivers. On the simulation level, one assumes certain statistical characteristics for the channel, but on a practical level, the actual channel is measured. In this letter we show that the accuracy of the measurements affects the accuracy of the capacity estimation when using the Shannon formula. We study analytically how the channel estimation error appears in the capacity formula, and we derive mathematical expressions for the first- and second-order approximations of the error. We also present simulation results that show the effect of the system size, the measurement accuracy, the system signal-to-noise ratio and the nature of the channel itself on the accuracy of the estimation of the channel capacity.**

***Index Terms*—Accuracy, capacity, channel estimation error, MIMO systems.**

I. Introduction: System Model and Notation

SYSTEMS with multiple transmitters and receivers can achieve very high spectral efficiencies, depending on the properties of the channel between them [1]. On the simulation level, one assumes certain statistical characteristics for the channel. For practical purposes, the actual channel is measured. The error in the channel measurement affects the calculation of the channel capacity.

Assume a system with M transmitters and N receivers. Without loss of generality we assume $M \leq N$. Each transmitter m sends an independent data stream x_m of power E_x so that the total power is $P_t(E_x = P_t/M)$. Let $\underline{x}$, $\underline{y}$ be the transmitted and received signal vectors respectively. In the case of a narrowband channel $\underline{y} = \mathbf{T}\underline{x} + \underline{n}$, where $\mathbf{T}$ is the channel transfer matrix (the element T_{ij} represents the complex channel gain from transmitter j to receiver i). $\underline{n}$ is the $(N \times 1)$-dimensional noise vector. Its components are assumed independent across the receivers, and each one of them is assumed to be additive white Gaussian noise of zero mean and variance σ^2.

The exact Shannon capacity is

$$C_{\text{exact}} = \log_2\left(\det\left(\mathbf{I}_N + \frac{E_x}{\sigma^2}\mathbf{T}\mathbf{T}^H\right)\right)$$
$$= \log_2\left(\det\left(\mathbf{I}_M + \frac{E_x}{\sigma^2}\left(\mathbf{T}^T\right)\left(\mathbf{T}^T\right)^H\right)\right)$$
$$= \sum_{i=1}^{K}\log_2\left(1 + \frac{E_x}{\sigma^2}\left|\lambda_i\right|^2\right). \quad (1)$$

$\mathbf{T}^H$ is for the complex conjugate transpose (Hermitian) of the matrix $\mathbf{T}$. λ_i is the i-th singular value of the matrix $\mathbf{T}$ (or $\mathbf{T}^T$). K is the number of nonzero singular values, i.e., the rank of the matrix $\mathbf{T}$. $K \leq \min(M, N)$.

The average channel gain g, the normalized channel transfer matrix $\mathbf{H}$ and the average signal to noise ratio (SNR) ρ are defined as

$$g^2 = E\left[\left|T_{ij}\right|^2\right], \mathbf{H} = \frac{1}{g}\mathbf{T}, \rho = g^2\frac{P_T}{\sigma^2}. \quad (2)$$

Then the channel capacity can be written as

$$C_{\text{exact}} = \log_2\left(\det\left(\mathbf{I}_N + \frac{\rho}{M}\mathbf{H}\mathbf{H}^H\right)\right). \quad (3)$$

Using the normalized channel transfer matrix $\mathbf{H}$ the channel capacity can be calculated for any reference SNR ρ_{ref} by substituting ρ_{ref} instead of ρ in (3).

If the measured value of $\mathbf{T}$ is $\mathbf{T}_{\text{meas}} = \mathbf{T} + \Delta\mathbf{T}$, then the normalized channel transfer matrix is $\mathbf{H}_{\text{meas}}$ and the channel capacity is estimated to be

$$C_{\text{meas}} = \log_2\left(\det\left(\mathbf{I}_N + \frac{E_x}{\sigma^2}\mathbf{T}_{\text{meas}}\mathbf{T}_{\text{meas}}^H\right)\right). \quad (4)$$

Similarly the capacity can then be calculated for any ρ_{ref}.

Our objective is to study the statistics of the error, $\varepsilon = C_{\text{meas}|\rho_{\text{ref}}} - C_{\text{exact}|\rho_{\text{ref}}}$, with respect to the system size, the channel characteristics, the estimation error and ρ_{ref}.

II. Error Calculation

Let $\mathbf{H}^T = \mathbf{S}\ \mathbf{U}\ \mathbf{V}^H$, $(\mathbf{H} = \mathbf{V}^*\ \mathbf{U}^T\ \mathbf{S}^T)$. The matrices $\mathbf{V}, \mathbf{S}$ are unitary, and the matrix $\mathbf{U}$ contains the singular values of $\mathbf{H}$. Let $\Delta\mathbf{H}^T = \mathbf{S}\Delta\mathbf{U}\ \mathbf{V}^H(\Delta\mathbf{U}$ is not diagonal!). Then

$$C_{\text{meas}} = \log_2\left(\det\left(\mathbf{I}_M + \frac{\rho_{\text{ref}}}{M}(\mathbf{U} + \Delta\mathbf{U})(\mathbf{U}^H + \Delta\mathbf{U}^H)\right)\right). \quad (5)$$

We compare with the actual capacity as follows:

$$\frac{2^{C_{\text{meas}}}}{2^{C_{\text{exact}}}} = \frac{\det\left(\mathbf{I}_M + \frac{\rho_{\text{ref}}}{M}(\mathbf{U} + \Delta\mathbf{U})(\mathbf{U}^H + \Delta\mathbf{U}^H)\right)}{\det\left(\mathbf{I}_M + \frac{\rho_{\text{ref}}}{M}\mathbf{U}\mathbf{U}^H\right)}$$

which yields

$$\frac{2^{C_{\text{meas}}}}{2^{C_{\text{exact}}}} = \det\left[\mathbf{I}_M + \rho_o\mathbf{G}\right] \quad (6)$$

where

$$\tilde{n}_0 = \frac{\tilde{n}_{\text{ref}}}{M}, \mathbf{Q} = \mathbf{I}_M + \frac{\tilde{n}_{\text{ref}}}{M}\mathbf{U}\mathbf{U}^H = \text{diag}\left(1 + \frac{\tilde{n}_{\text{ref}}}{M}\left|u_i\right|^2\right)$$
$$\mathbf{G} = (\ddot{A}\mathbf{U}\mathbf{U}^H + \mathbf{U}\ddot{A}\mathbf{U}^H + \ddot{A}\mathbf{U}\ddot{A}\mathbf{U}^H\mathbf{Q}^{-1}. \quad (7)$$

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If the matrices $\mathbf{U}$ and $\Delta\mathbf{U}$ are partitioned into two matrices, one of dimension $M \times M$ and one of dimension $M \times (N - M)$, such that $\mathbf{U} = [\mathbf{U}_0 \quad \mathbf{0}]$, $\Delta\mathbf{U} = [\mathbf{U}_I \quad \mathbf{U}_{II}]$, we can write

$$\mathbf{G} = (\mathbf{U}_I\mathbf{U}_0^* + \mathbf{U}_0\mathbf{U}_I^H + \mathbf{U}_{II}\mathbf{U}_{II}^H + \mathbf{U}_{II}\mathbf{U}_{II}^H)\mathbf{Q}^{-1}. \tag{8}$$

Let $L = N - M$, and $\mathbf{U}_0 = \mathrm{diag}(u_i)$, $(\mathbf{U}_I)_{ij} = u_{ij}^I$, $(\mathbf{U}_{II})_{ij} = u_{ij}^{II}$. Using this notation

$$G_{ij} = \frac{u_j^* u_{ij}^I + u_i u_{ij}^{I^*} + \sum_{k=1}^{M} u_{ik}^I u_{jk}^{I^*} + \sum_{k=1}^{L} u_{ik}^{II} u_{jk}^{II^*}}{1 + \rho_0 |u_j|^2}. \tag{9}$$

We calculate the determinant of the matrix $\mathbf{I}_M + \rho_0\mathbf{G}$ and keep only the first- and second-order terms, as in

$$\frac{2^{C_{\mathrm{meas}}}}{2^{C_{\mathrm{exact}}}} = 1 + x_1 + x_2 + x_3 - x_4 \tag{10}$$

$$x_1 = \sum_{j=1}^{M} \rho_0 \frac{u_j^* u_{jj}^I + u_j u_{jj}^{I^*}}{1 + \rho_0 |u_j|^2} \tag{11}$$

$$x_2 = \sum_{j=1}^{M} \rho_0 \frac{\sum_{k=1}^{M} \left|u_{jk}^I\right|^2 + \sum_{k=1}^{L} \left|u_{jk}^{II}\right|^2}{1 + \rho_0 |u_j|^2} \tag{12}$$

$$x_3 = \sum_{j=1}^{M} (\rho_0)^2 \sum_{k=1, k\neq j}^{M} \frac{(u_j^* u_{jj}^I + u_j u_{jj}^{I^*})}{1 + \rho_0 |u_j|^2} \frac{(u_k^* u_{kk}^I + u_k u_{kk}^{I^*})}{1 + \rho_0 |u_k|^2} \tag{13}$$

$$x_4 = -\sum_{j=1}^{M} (\rho_0)^2 \sum_{k=1, k\neq j}^{M} \frac{(u_k^* u_{kj}^I + u_j u_{jk}^{I^*})}{1 + \rho_0 |u_k|^2} \frac{(u_j^* u_{jk}^I + u_k u_{kj}^{I^*})}{1 + \rho_0 |u_j|^2}. \tag{14}$$

The minus sign in the last term results from the permutation.

The error gets projected onto the eigenvectors of the matrix $\mathbf{H}$. The error along the input eigenvectors of $\mathbf{H}^T$ (output eigenvectors of $\mathbf{H}$) that correspond to zero eigenvalues affects less than the error made along the vectors that correspond to nonzero eigenvalues.

Ignoring the terms of third order and higher, and removing the logarithm

$$\begin{aligned} \varepsilon &= C_{\mathrm{meas}} - C_{\mathrm{exact}} = \log_2(1 + x_1 + x_2 + x_3 + x_4) \\ &\approx \frac{1}{\ln 2}\left[x_1 + \left(x_2 + x_3 + x_4 - \frac{1}{2}x_1^2\right)\right]. \end{aligned} \tag{15}$$

III. Statistics of the Error

A. Statistics of the First-Order Error

The first order error is

$$\varepsilon_1 = \frac{1}{\ln 2} x_1. \tag{16}$$

The error in the estimation of the element h_{ij} of the matrix $\mathbf{H}$ is a complex number of the form $\Delta h_{ij} = x_{ij} + jy_{ij}$. If x_{ij}, y_{ij} for any i, j are independent, identically Gaussian distributed random variables with zero mean and variance $\sigma_m^2/2$, then the first order error is the sum of scaled Gaussian random variables and is itself a Gaussian random variable with zero mean and variance

$$\sigma_{\varepsilon_t}^2 = \frac{1}{(\ln 2)^2} \sum_{i=1}^{M} 2(\rho_0)^2 \frac{|u_i|^2}{(1 + \rho_o |u_i|^2)^2} \sigma_m^2. \tag{17}$$

B. Statistics of the Second-Order Error

The second order error is not Gaussian distributed, and its mean value is not zero.

$$\varepsilon_2 = \frac{1}{\ln 2}\left(x_2 + x_3 + x_4 - \frac{1}{2}x_1^2\right). \tag{18}$$

IV. Simulation Results

Assume that the error in the estimation of the channel transfer matrix can be written in the form $\Delta\mathbf{T} = \sigma_H \mathbf{R}$, where $\mathbf{R}$ is a matrix with independent complex Gaussian distributed random entries with variance 1. The capacity estimate is shown in

$$\begin{aligned} C &= \log_2\left(\det\left(\mathbf{I}_N + \frac{\rho_{\mathrm{ref}}}{M}\mathbf{H}_{\mathrm{meas}}\mathbf{H}_{\mathrm{meas}}^H\right)\right) \\ &= \log_2\left(\det\left(\mathbf{I}_N + \frac{\rho_{\mathrm{ref}}}{M}\left(\frac{1}{g_{\mathrm{meas}}}\mathbf{T}_{\mathrm{meas}}\right)\left(\frac{1}{g_{\mathrm{meas}}}\mathbf{T}_{\mathrm{meas}}\right)^H\right)\right). \end{aligned} \tag{19}$$

If $\sigma_H \ll g$, then see

$$g_{\mathrm{meas}}^2 = E\left[|T_{\mathrm{meas},ij}|^2\right] \approx g^2 + \sigma_H^2, \frac{1}{g_{\mathrm{meas}}} = \frac{1}{g\sqrt{1 + \frac{\sigma_H^2}{g^2}}} \approx \frac{1}{g}$$

$$C = \log_2\left(\det\left(\mathbf{I}_N + \frac{\rho_{\mathrm{ref}}}{M}\left(\mathbf{H} + \frac{1}{\sqrt{\rho_H}}\mathbf{R}\right)\left(\mathbf{H} + \frac{1}{\sqrt{\rho_H}}\mathbf{R}\right)^H\right)\right). \tag{20}$$

The relative size of the error with respect to the average channel gain is given by the measurement SNR ρ_H. For example in a system where the channel estimation is done with orthogonal training sequences as in [2], ρ_H depends on the length of the training sequence.

$$\rho_H = \frac{g^2}{\sigma_H^2} \quad . \tag{21}$$

In the following simulations we limit ourselves to square $M \times M$ systems. Obviously the error statistics depend on the actual normalized channel transfer matrix $\mathbf{H}$. We focus on two limiting cases: The best-case scenario is one that maximizes the channel capacity. This occurs when the channel transfer matrix has M equal singular values, i.e., the channel is equivalent to M parallel sub-channels all of the same gain. Any orthonormal matrix would satisfy this condition so we select the identity matrix. The worst-case scenario is the one that minimizes the channel capacity. This occurs when the channel transfer matrix has a single eigenmode, i.e., only one nonzero eigenvalue. A matrix that satisfies this condition is one with all elements equal. So

$$\mathbf{H}_{\mathrm{best\ case}} = \sqrt{M}\mathbf{I}, (\mathbf{H}_{\mathrm{worst\ case}})_{ij} = 1. \tag{22}$$

The elements of both matrices are appropriately scaled so that $E[|H_{ij}|^2] = 1$.

A. Effect of the Measurement SNR ρ_H

Consider a 4×4 system, $\rho_{\mathrm{ref}} = 20$ dB. Fig. 1 shows the cumulative distribution function of the estimated capacity over several realizations of the random matrix $\mathbf{R}$, as above.

The higher ρ_H, the better the estimate of the capacity. The error for the best-case scenario is negligible. In the worst-case scenario, measurement error leads to an overestimation of the actual capacity. This is due to the excitation of eigenmodes that the true channel would not excite.

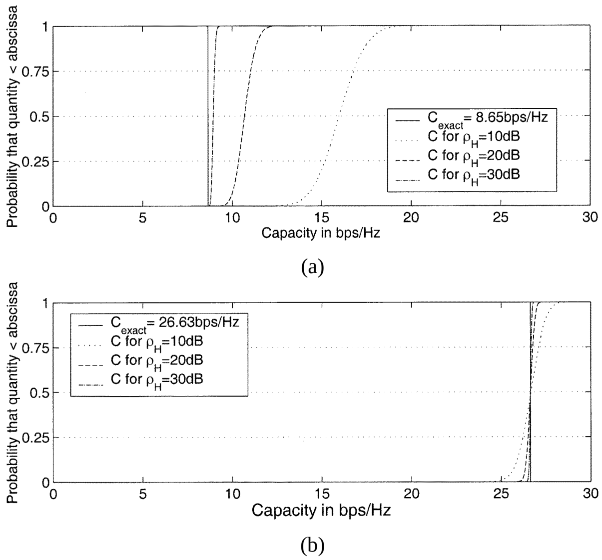

Fig. 1. CDF of estimated capacity. (a) Worst-case scenario. Capacity for $\rho_{\mathrm{ref}} = 20$ dB. (b) Best-case scenario. Capacity for $\rho_{\mathrm{ref}} = 20$ dB.

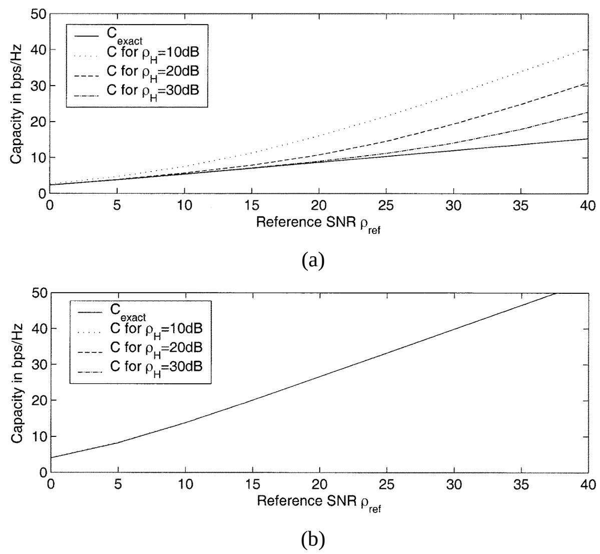

Fig. 2. Effect of reference SNR ρ_{ref}. (a) Worst-case scenario. Capacity for various measurement SNRs. (b) Best-case scenario. Capacity for various measurement SNRs.

B. Effect of the Reference SNR ρ_{ref}

Assume that several realizations of the channel transfer matrix are recorded, and the capacity is calculated for each of them. The estimated capacity is taken as the mean over these realizations, i.e., as the mean of CDF curves similar to those shown in Fig. 1. Fig. 2 shows how the estimated channel capacity varies with ρ_{ref} for a 4×4 system.

For the best-case scenario, the capacity estimate is accurate for a wide range of measurement and reference signal-to-noise ratios. However for the worst-case scenario, the error increases as either the reference SNR ρ_{ref} increases or the measurement SNR ρ_H decreases, and it can become of the order of 100%.

C. Effect of the System Size

Fig. 3 shows how the capacity estimation error varies with the number of antennas M (at both the transmitter and the receiver) for $\rho_{\mathrm{ref}} = 20$ dB and variable ρ_H.

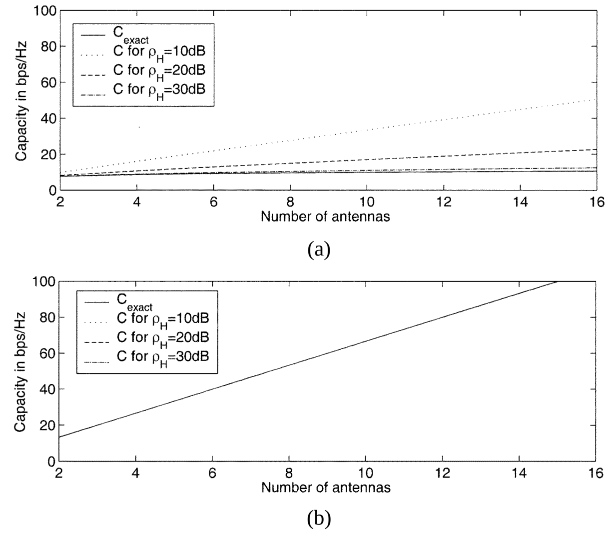

Fig. 3. Effect of number of antennas. (a) Worst-case scenario. Capacity for various numbers of antennas. (b) Best-case scenario. Capacity for various numbers of antennas.

Capacity grows linearly with the number of transmitters for the best-case scenario and logarithmically for the worst-case scenario. The error is negligible for the best-case scenario for any system size, but it grows with the number of antennas for the worst-case scenario.

V. Conclusions

We investigated how the error in the channel estimation affects the calculation of the channel capacity, as per the Shannon formula.

One can derive an analytical expression for the first and second order error. The capacity estimation error depends on the accuracy of the channel measurement, the reference SNR and the nature of the channel itself.

We studied two limiting cases for the channel. For a real-life system the error lays between these worst- and best-case scenarios. In the best-case scenario (independent channels of equal gains), the error is very small independently of the system size or ρ_H. In the worst-case case scenario the multiple antenna system reduces to a single effective channel. The error grows with the increasing system size, increasing ρ_{ref} and with decreasing ρ_H.

There are essentially four parameters: the system size, ρ_{ref}, ρ_H and the desired accuracy. If three of these parameters are set, then the fourth is uniquely specified. This is a useful pointer for system designers, when specifying the parameters of their measurement campaign (transmit power, range, etc).

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