Estimating Local Mean Signal Power Level in a Rayleigh Fading Environment

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***Abstract*—Good estimates of local mean signal power level are needed by wireless cellular systems for use in channel access, power control and handoff. Many systems today use weighted sample average estimators which are not optimal in Rayleigh fading environments. The optimal local mean signal level estimator for the Rayleigh fading environment will be derived and compared with the sample average estimator.**

***Index Terms*—Fading, parameter estimation.**

I. Introduction

IN WIRELESS cellular communications systems, it is very important to have an accurate estimate of local mean signal power level. Local mean signal level is a good indicator of communication link quality and is used in system functions like channel access, handoff, and power control. The more accurately a system estimates local mean signal level, the more efficiently these functions are performed.

Small-scale fading, occurring as local fluctuations around the local mean signal level, is often modeled as a random process, where the signal magnitude is Rayleigh distributed with mean given by the local mean signal level. In digital systems, the signal level measurements are normally discrete-time real-valued samples in decibels. The samples are in decibels because the output of the amplifiers often has a logarithmic characteristic and because this allows for a wide dynamic range of power levels. We wish to estimate the local mean signal level by finding the mean of the samples. An estimate of the signal level in decibels is desired, rather than the decibel value of an estimate of average linear power because it is the values in decibels which are often used in the various system functions like channel access, power control, and handoff. Suppose that Y were Rayleigh distributed. Then, the distribution of $X = 20 \log Y$ would be what we will call antilog Rayleigh (ALR). The probability density function of the ALR distribution is given by

$$f_x(x) = (\ln\ 10)\frac{10^{x/10}}{20p} \exp\left(-\frac{10^{x/10}}{2p}\right) \tag{1}$$

where p is a power parameter given by $2p = E[Y^2]$.

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Let $\{X_j\}$ be a set of N independent identically distributed (IID) ALR random variables with parameter p. We want an unbiased estimate of the mean, with minimum mean-square error (equivalent to a minimum variance unbiased estimate). It is assumed that all N samples have the same mean because local mean varies slowly over short distances (a few tens of meters). It is also assumed that the user moves at reasonable speeds, so successive samples are far apart enough to be uncorrelated (e.g., about 30 cm at 900 MHz).

Many systems like GSM [1][1] and personal access communication system (PACS) [2] employ a weighted sample average estimator. Many papers on handoff assume that when sample averaging is used, the effects of small-scale fading can be ignored in analyzing handoff algorithms (e.g., [3] and [4]). This assumption is incorrect because only an estimate of the local mean is thus obtained. The performance of both continuous-time average estimators and sample average estimators have been previously investigated [5], [6], but we know of no publication of either a derivation or an examination of optimum minimum variance unbiased estimators for this problem.

II. Analysis

The performance of the sample-average estimator

$$E_{sa} = \frac{1}{N} \sum_{j=1}^{N} X_j \tag{2}$$

will be compared with that of the optimum (minimum variance) unbiased estimator

$$E_{op} = 10\left[\log T - \frac{H_{N-1}}{\ln\ 10}\right] \tag{3}$$

(where $T = \Sigma_{j=1}^{N} 10^{X_j/10}$ and $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + (1/n)$ for $n = 1, 2, \cdots$). H_n is also known as the nth *harmonic number*. E_{op} is derived in Appendix A.

A. Performance Comparisons

E_{sa} is unbiased and consistent, but not minimum variance. Its variance is

$$\mathrm{Var}(E_{sa}) = \frac{1}{N}\left(\frac{50\pi^2}{3(\ln\ 10)^2}\right).$$

If X is ALR distributed with parameter p, then using (1), $E[X] = 10[\log(2p) - \gamma/(\ln\ 10)]$, where $\gamma \approx 0.577\,216$

[1] GSM originally stood for Groupe Special Mobile, and today stands for Global System for Mobile Communications.

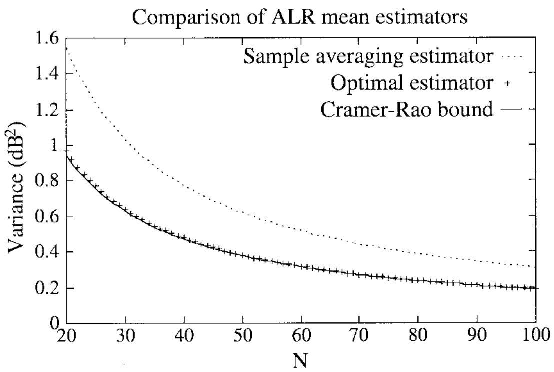

Fig. 1. Variance (or MSE) of the minimum variance unbiased estimator of ALR mean, compared with that of the sample averaging estimator, and with the Cramér–Rao bound.

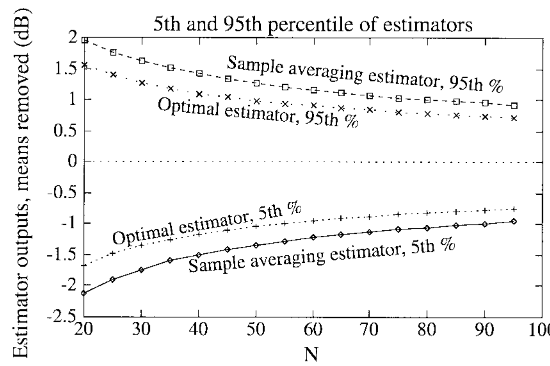

Fig. 2. The 5th and 95th percentiles of the estimators obtained by computer simulation. The means have been removed, so these results are applicable for all positive values of p because changing p is equivalent to a shift in the distribution of the estimators.

is Euler's gamma constant. So E_{op} is easily verified to be unbiased, since $E[\log\ T] = (H_{N-1} - \gamma)/(\ln\ 10) + \log 2p$. Furthermore

$$\mathrm{Var}(E_{op}) = \frac{100}{(\ln\ 10)^2}\left[\frac{\pi^2}{6} - \sum_{k=1}^{N-1}\frac{1}{k^2}\right]. \tag{4}$$

A lower bound to the variance of any unbiased estimator is the Cramér–Rao bound [7]. The Cramér–Rao lower bound for our problem is $100/[N(\ln\ 10)^2]$, which is shown with E_{op} and E_{sa} in Fig. 1. Note that the Cramér–Rao bound is not necessarily an achievable lower bound. In fact, it can be easily shown to not be achievable by any unbiased estimator for our problem. We will show, however, that E_{op} *asymptotically* achieves the Cramér–Rao bound, as a corollary to a proof about the *improvement ratio function*.

We define the improvement ratio function $R_{\mathrm{ALR}}(N)$ as

$$R_{\mathrm{ALR}}(N) = \frac{\mathrm{Var}(E_{sa})}{\mathrm{Var}(E_{op})} = \frac{\pi^2}{N\left(\pi^2 - 6\displaystyle\sum_{k=1}^{N-1} 1/k^2\right)}. \tag{5}$$

$R_{\mathrm{ALR}}(N)$ can be shown to be monotonically increasing with N, and $N \geq 15$ yields $R_{\mathrm{ALR}}(N) > 1.6$, approaching $\pi^2/6$ for large N. So for any given N, the mean-square error of E_{sa} is about 1.65 times larger than that of E_{op}. Confidence intervals for E_{op} are consequently also shorter than the corresponding confidence intervals for E_{sa}.

To show that $R_{\mathrm{ALR}}(N) \to \pi^2/6$, (4) can be written as $\mathrm{Var}(E_{op}) = [100/(\ln\ 10)^2]\ \Sigma_{k=N}^{\infty}\ 1/k^2$ using the identity $\Sigma_{k=1}^{\infty}\ 1/k^2 = \pi^2/6$. Using appropriate continuous integrals to approximate the summation, lower and upper bounds can be found

$$\frac{100}{(\ln\ 10)^2 N} < \frac{100}{(\ln\ 10)^2}\sum_{k=N}^{\infty}\frac{1}{k^2} < \frac{100}{(\ln\ 10)^2(N-1)}.$$

It turns out that this lower bound is also the Cramér–Rao lower bound, and since the upper and lower bounds approach each other as $N \to \infty$, E_{op} asymptotically achieves the Cramér–Rao lower bound as $N \to \infty$. Therefore, $R_{\mathrm{ALR}}(N) \to \pi^2/6$.

The variances of the estimators do not depend on p and neither does their ratio. So these performance improvements are applicable over the wide range of p that may be found in a wireless system.

B. Practical Implications

The mean-square error obtained with E_{op} is consistently lower (better) than that obtained with E_{sa}. E_{sa}, however, is simple and not as distribution dependent as E_{op}, which works best with ALR random variables. It is also easier to implement, possibly even without the use of floating point arithmetic. E_{op}, however, might be useful in environments in which the small-scale fading is strongly Rayleigh distributed, e.g., a high-power cellular mobile system operating in the downtown area of a city.

How many samples are needed before the estimators meet certain quality criteria? Consider, for example, the number of samples that are needed such that the estimated mean has a standard deviation below 1 dB, equivalent to the point where it comes within 1 dB of the true mean with probability 68.27%, for Gaussian distributions (NB: both estimators have distributions which approach Gaussian for large enough N). From Fig. 1, about 31 samples are needed for E_{sa} to achieve this, while only 19 samples are needed for E_{op}. As a second example, 5th and 95th percentiles are considered. These are shown in Fig. 2. Approximately 50 samples are needed for the difference between the two percentile lines to be 2 dB for E_{op}. About 85 samples are needed for similar performance with E_{sa}.

The number of samples that are needed to achieve certain degrees of estimator accuracy is critical because of the presence of shadow fading. The N samples will not all have the same mean, as assumed in this paper, but the mean will be slowly changing within the measurement window. This will cause the variance of the estimators to increase. So we do not want N to be too large (for example, over 50). Therefore, N cannot be rashly increased to arbitrarily reduce the variance. Since N is limited by design considerations (based on shadow fading variance and other things), it is desirable that for a given N the estimator performs as well as it can.

It is to be emphasized that E_{op} is designed for use where the small-scale fading is Rayleigh distributed. However, the case where there is a dominant path component has also been investigated by simulation, where the small-scale fading is modeled as Ricean distributed, and the mean power of the dominant component is p_d and the mean power from the rest of the paths is $2p$. The critical factor is $k = p_d/2p$. For $k < 0.5$, the performance of E_{op} is practically unchanged, with just a slight bias introduced as k approaches 0.5. Even as k increases to one and larger, the variance of E_{op} is consistently smaller than that of E_{sa}, but for $0.5 < k < 1$, the mean-square error of E_{op} starts increasing significantly (as the bias increases), and continues to do so for $k > 1$, where eventually the mean-square error of E_{op} becomes larger than that of E_{sa}. So E_{op} should be used only where the small-scale fading is Rayleigh distributed or mildly Ricean distributed. However, it is to be noted that even in Ricean fading environments, the bias is in the direction to downplay the contribution of the dominant component, which is in the right direction, if any, given that the dominant component can rapidly change (e.g., if the user turns around a corner).

III. Concluding Remarks

The optimum minimum variance unbiased mean estimator for ALR-distributed random variables was found. Comparisons were made with traditional sample average estimators. It was found that significantly fewer samples are required for a given estimation accuracy when using the optimum estimator. The estimator of choice in some cellular communications standards (such as PACS and GSM) is the sample average estimator, or a variant thereof. Because of the importance of obtaining an accurate, low-variance estimate of local mean signal power before that mean changes, more optimal estimators, such as E_{op}, may be desirable in future standards.

Appendix A

We derive the minimum-variance unbiased estimator of the mean of N IID ALR-distributed random variables.

Let $\boldsymbol{X} = \{X_1, X_2, \cdots, X_N\}$, and the $X_j \sim (\ln 10)(10^{x/10}/20p)\exp(-(10^{x/10}/2p)$, $x > 0$, for $j = 1, 2, \cdots, N$ are the IID components of the vector $\boldsymbol{X}$. Then the mean of each of the components of $\boldsymbol{X}$ is $\mu = 10[\log(2p) - \gamma/(\ln\ 10)]$.

It can be shown (for example, by using likelihood ratios) that a minimal sufficient statistic for $\boldsymbol{X}$ is the likelihood statistic

$$T(\boldsymbol{X}) = \sum_{j=1}^{N} 10^{X_j/10}.$$

$T(\boldsymbol{X})$ is also a complete sufficient statistic[2] because the parameter space of μ is one-dimensional (1-D), and the family of ALR distributions obtained by varying μ is a one-parameter exponential type, which implies completeness [8].

We are now ready to use the following theorem.

[2] A family of distribution functions $\mathcal{F} = \{f_\theta(x); \theta \in \Theta\}$ is complete if for any integrable function $g(x)$

$$\int g(x) f_\theta(x)\ dx = 0 \quad \text{for all } \theta \in \Theta\ .$$

It implies that $P_\theta[g(x) = 0] = 1$ for all $\theta \in \Theta$.

Theorem 1: **Rao–Blackwell Lehmann–Scheffé Theorem**[3]: Let $\mathcal{F} = \{F(\boldsymbol{X};\theta); \theta \in \Theta\}$ be a family of vector random variables as described before. Suppose that $w = g(\theta)$ is a parameter having an unbiased estimator $\hat{g}(\boldsymbol{X})$. If there exists a minimal sufficient statistic $T(\boldsymbol{X})$, then $\hat{w} = E\{\hat{g}(\boldsymbol{X})|T(\boldsymbol{X})\}$ is an unbiased estimator of w and $\mathrm{Var}_\theta\{\hat{w}\} \le \mathrm{Var}_\theta\{\hat{g}(\boldsymbol{X})\}$ for all $\theta \in \Theta$. In addition, if $T(\boldsymbol{X})$ is a complete sufficient statistic, then $\hat{w}$ is the unique minimum variance unbiased estimator, for each $\theta \in \Theta$.

For the present derivation, w corresponds to μ. Select $\hat{g}(\boldsymbol{X}) = X_{j_0}$ for some $j_0, 1 \le j_0 \le N$. By Theorem 1, the desired unique minimum variance unbiased estimator is

$$\hat{w} = E\{\hat{g}(\boldsymbol{X})|T(\boldsymbol{X})\}. \tag{6}$$

$T(\boldsymbol{X})$ is chi-squared distributed with $2N$ degrees of freedom. So the joint probability distribution is

$$\begin{aligned} &p_{X_{j_0}, T(\boldsymbol{X})}(x, T) \\ &\quad = (\ln\ 10)\frac{10^{x/10}}{20p}\exp\left(-\frac{10^{x/10}}{2p}\right)\frac{1}{p^{N-1}2^{N-1}(N-2)!} \\ &\quad \cdot (T - 10^{x/10})^{N-2}\exp\left(-\frac{T - 10^{x/10}}{2p}\right), \\ &\quad\quad \text{for } x < 10\ \log\ T. \end{aligned} \tag{7}$$

Hence, the conditional probability distribution is

$$\begin{aligned} &p_{X_{j0}|T(\boldsymbol{x})}(x|T) \\ &\quad = \frac{\ln(10)10^{x/10}(N-1)}{10T}\left(1 - \frac{10^{x/10}}{T}\right)^{N-2}, \\ &\quad\quad \text{for } -\infty < x < 10\ \log\ T. \end{aligned} \tag{8}$$

The optimal estimator is then given by

$$\begin{aligned} E_{op} &= E[X_{j0}|T] = \int_{-\infty}^{10\log T} \frac{(N-1)x\ \ln(10)^{x/10}}{10T} \\ &\quad \cdot \left(1 - \frac{10^{x/10}}{T}\right)^{N-2} dx \\ &= \frac{10(N-1)}{\ln\ 10}\left(\sum_{i=0}^{N-2} \binom{N-2}{i}\right. \\ &\quad \left. \cdot \left(\frac{\ln\ T}{i+1} - \frac{1}{(i+1)^2}\right)(-1^i\right) \end{aligned} \tag{9}$$

where the last equality is obtained by changing the variable of integration to $y = 10^{x/10}$ and integrating term-by-term using the identity

$$\int_0^T (\ln\ y)y^i\ dy = \frac{T^{i+1}}{(i+1)}\left(\ln\ T - \frac{1}{i+1}\right).$$

It can be shown that for $N \ge 2$

$$(N-1)\sum_{i=0}^{N-2} \binom{N-2}{i}\frac{1}{(i+1)^2}(-1)^i = H_{N-1}$$

[3] A proof of this theorem is found in [8].

and so

$$E_{op} = 10\left[\log T - \frac{H_{N-1}}{\ln\ 10}\right], \quad \text{where} \quad T = \sum_{j=1}^{N} 10^{X_j/10}.$$

APPENDIX B

Similar analysis can be done for the corresponding Rayleigh problem, i.e., when the ALR-distributed random variables are replaced with Rayleigh-distributed ones. This might have practical application for systems which take sample averages of absolute signal magnitude samples, rather than decibel samples. The likelihood statistic, which is also a complete minimal sufficient statistic, in this case is

$$T() = \sum_{j=1}^{N} X_j^2 \tag{10}$$

where the same notation is used, except that now $\boldsymbol{X}$ is a random vector of IID Rayleigh random components, rather than ALR components. After going through a derivation similar to that for the ALR problem, the optimum (minimum-variance) unbiased estimator is found to be

$$\int_0^{\sqrt{T}} \frac{2(N-1)x^2}{T}\left(1 - \frac{x^2}{T}\right)^{N-2} dx = \frac{(N-1)!\sqrt{\pi T}}{2\Gamma(N+1/2)} \tag{11}$$

where $T = \Sigma_{j=1}^{N} X_j^2$ and $\Gamma(\cdot)$ is the gamma function. There is a performance improvement, but it is not as much in this case as it is in the ALR case.

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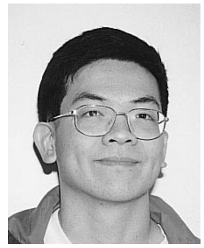

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